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Extremal properties of the univariate extended skew-normal distribution, Part A

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ABSTRACT

We consider the extremal properties of the highly flexible univariate extended skew-normal distribution. We derive the well-known Mills' inequalities and Mills' ratio for the extended skew-normal distribution and establish the asymptotic extreme-value distribution for the maximum of samples drawn from this distribution.

1. Introduction and background

The skew-normal and related families are classes of asymmetric probability distributions that include the normal distribution as a special case (Azzalini, 1985; Azzalini and Capitanio, 2014). In recent years they have received increasing interest from the scientific community because in many applications data is frequently incompatible with symmetric distributions, such as the normal or elliptical distributions.

In this work we focus on the univariate extended skew-normal distribution (Arellano-Valle and Genton, 2010; Azzalini and Capitanio, 2014, Ch 5.). Precisely, a random variable X follows an extended skew-normal distribution (Arellano-Valle and Genton, 2010), denoted as $X \sim ESN(\mu, \omega, \alpha, \tau)$, if its probability density function (pdf) is given by

$$\phi(x;\mu,\omega,\alpha,\tau) = \frac{\phi(x;\mu,\omega)}{\Phi\left(\tau/\sqrt{1+\alpha^2}\right)}\Phi(\alpha z+\tau), \quad x \in \mathbb{R}, \tag{1}$$

where $\phi(x;\mu,\omega)$ is a univariate normal pdf with mean $\mu \in \mathbb{R}$ and standard deviation $\omega > 0$, $z = \omega^{-1}(x-\mu)$, and $\Phi(\cdot)$ is the standard univariate normal cumulative distribution function (cdf). The parameters $\alpha \in \mathbb{R}$ and $\tau \in \mathbb{R}$ are known as the slant and extension parameters, respectively, and they control the nature of density deviations away from normality. When $\tau = 0$ the extended skew-normal distribution reduces to the skew-normal $SN(\mu,\omega,\alpha)$, and when both $\tau = 0$ and $\alpha = 0$ the normal $N(\mu,\omega)$ distribution is obtained. Without loss of generality, we work with location and scale standardised distributions throughout, i.e. $\mu = 0$ and $\omega = 1$ which we compactly denote as $ESN(\alpha,\tau)$. For further visual and presentational clarity we write distributional parameters in the subscript of the pdf and cdf so that e.g. $\phi(x;\alpha,\tau) = \phi_{\alpha,\tau}(x)$ and $\Phi(x;\alpha,\tau) = \Phi_{\alpha,\tau}(x)$.

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Our contribution concerns the derivation of the extremal properties of the univariate extended skew-normal distribution. Specifically, we obtain the well-known Mills' inequalities and ratio (Mills, 1926), and as a result, derive the asymptotic extreme-value distribution for the maximum of an extended skew-normal random sample, for large sample sizes. The speed at which the sample distribution converges to its limiting case is also determined.

We briefly recall the cornerstone result of univariate extreme-value theory. For each $n \in \mathbb{N}$, let $X_1, \ldots, X_n$ be a series of independent and identically distributed (*iid*) univariate random variables with a continuous distribution function F defined on $\mathbb{R}$. Define, the (n-partial) sample maximum by $M_n = \max(X_1, \ldots, X_n)$. If there is a sequence of normalising constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \tag{2}$$

for all continuity points $x \in \mathbb{R}$ of G, then this limiting distribution must be a member of the Generalised Extreme-Value (GEV) family of distributions, denoted by G_γ where $\gamma \in \mathbb{R}$ is the tail-index parameter (Beirlant et al., 2004, Section 2.1). Specifically, members of the GEV class of distributions are: the standard Gumbel $G_0(x)$, Fréchet $G_\alpha(x) = G_{1/\gamma}((x-1)/\gamma)$ for $\gamma > 0$ and negative Weibull $G_\beta(x) = G_{-1/\gamma}(-(x+1)/\gamma)$ for $\gamma < 0$. When the limit (2) holds we say that F is in the maximum-domain of attraction of G_γ, in symbols $F \in \mathcal{D}(G_\gamma)$. One well known result is that a necessary and sufficient condition for $F \in \mathcal{D}(G_\gamma)$ is that F is a Von Mises function, see Resnick (1987, Ch. 1) and Resnick (1987, Proposition 1.4) for the special case when $F \in \mathcal{D}(G_0)$.

The remainder of this paper is organised as follows. In Section 2 we firstly derive Mills' inequalities and ratio, and then the extreme-value distribution for the sample maximum and the convergence rate of its sample distribution. Throughout, all proofs are provided in the Appendix.

2. Extremes of extended skew-normal random samples

Mills (1926) established the following results (Mill's inequalities and Mills' ratio) for the standard normal distribution:

$$x^{-1}(1+x^{-2})^{-1} < \frac{1-\Phi(x)}{\phi(x)} < x^{-1}, \quad x > 0, \tag{3}$$

$$\frac{1-\Phi(x)}{\phi(x)} \approx x^{-1}, \quad x \to \infty, \tag{4}$$

where (4) is obtained from (3) for large x. Mills' ratio can be used to establish the normalising constants a_n and b_n in (2) following Proposition 1.1 in Resnick (1987). Liao et al. (2014) derived Mills' inequalities and ratio for the skew-normal distribution, from which (3) and (4) may be recovered by setting $\alpha = 0$. Here we require more general results. The following two propositions derive Mills' inequalities (Proposition 2.1) and ratio (Proposition 2.2) for the extended skew-normal distribution. It follows that the results in Liao et al. (2014) can be obtained from these by setting $\tau = 0$.

Proposition 2.1 (*Mills' Inequality*). *Let $X \sim ESN(\alpha, \tau)$ where $\alpha, \tau \in \mathbb{R}$. For each $x \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}$, define $x_{\alpha,\tau} := \alpha x + \tau$ and $\bar{\alpha} = (1+\alpha^2)^{1/2}$. For any $x > 0$ we have*

$$L_{\alpha,\tau}(x) < \frac{1-\Phi_{\alpha,\tau}(x)}{\phi_{\alpha,\tau}(x)} < U_{\alpha,\tau}(x),$$

where the upper and lower bounds are given as follows:

(i) when $\alpha \geq 0$

(ia) and when $x_{\alpha,\tau} > 0$, then

$$L_{\alpha,\tau}(x) = x^{-1}\left(1+x^{-2}\right)^{-1} \quad \text{and} \quad U_{\alpha,\tau}(x) = x^{-1}\left(1 - \tfrac{\phi(x_{\alpha,\tau})}{x_{\alpha,\tau}}\right)^{-1},$$

(ib) and when $x_{\alpha,\tau} < 0$, then

$$L_{\alpha,\tau}(x) = x^{-1}\left(1+x^{-2}\right)^{-1} \quad \text{and} \quad U_{\alpha,\tau}(x) = x^{-1}\left(-\tfrac{x_{\alpha,\tau}^2+1}{x_{\alpha,\tau}\phi(x_{\alpha,\tau})}\right),$$

(ii) when $\alpha < 0$

(iia) and when $x_{\alpha,\tau} > 0$ and $x + \alpha x_{\alpha,\tau} > 0$, then

$$L_{\alpha,\tau}(x) = x^{-1}\left(1+x^{-2}\right)^{-1}\left(1 + \frac{\alpha}{\bar{\alpha}^2 x + \alpha\tau}\frac{\phi(x_{\alpha,\tau})x_{\alpha,\tau}}{x_{\alpha,\tau} - \phi(x_{\alpha,\tau})}\right),$$

$$U_{\alpha,\tau}(x) = x^{-1}\left(1 + \frac{\alpha(\bar{\alpha}^2 x + \alpha\tau)}{(\bar{\alpha}^2 x + \alpha\tau)^2 + \bar{\alpha}^2}\frac{\phi(x_{\alpha,\tau})(\bar{\alpha}^2 x + \alpha\tau)}{x_{\alpha,\tau}^2 + 1 - x_{\alpha,\tau}\phi(x_{\alpha,\tau})}\right),$$

(iib) and when $x_{\alpha,\tau} > 0$ and $x + \alpha x_{\alpha,\tau} < 0$, then

$$L_{\alpha,\tau}(x) = x^{-1}\left(1+x^{-2}\right)^{-1}\left(1+\frac{\frac{\alpha}{\bar\alpha}\phi\left(\bar\alpha x+\frac{\alpha\tau}{\bar\alpha}\right)^{-1}+\frac{\alpha\{\bar\alpha^2 x+\alpha\tau\}}{\{\bar\alpha^2 x+\alpha\tau\}^2+\bar\alpha^2}}{\phi(x_{\alpha,\tau})^{-1}-x_{\alpha,\tau}^{-1}}\right),$$

$$U_{\alpha,\tau}(x) = x^{-1}\left(1+\frac{\frac{\alpha}{\bar\alpha}\phi\left(\bar\alpha x+\frac{\alpha\tau}{\bar\alpha}\right)^{-1}+\frac{\alpha}{\bar\alpha^2 x+\alpha\tau}}{\phi(x_{\alpha,\tau})^{-1}-\frac{x_{\alpha,\tau}}{x_{\alpha,\tau}^2+1}}\right),$$

(iic) and when $x_{\alpha,\tau} < 0$, then

$$L_{\alpha,\tau}(x) = x^{-1}\left(1+x^{-2}\right)^{-1}\left\{1-\frac{\alpha x_{\alpha,\tau}}{\bar\alpha^2 x+\alpha\tau}\left(1+\frac{1}{x_{\alpha,\tau}^2}\right)\right\},$$

$$U_{\alpha,\tau}(x) = x^{-1}\left(1-\frac{\alpha x_{\alpha,\tau}}{\bar\alpha^2 x+\alpha\tau}\left(1+\frac{\bar\alpha^2}{(\bar\alpha^2 x+\alpha\tau)^2}\right)^{-1}\right).$$

Proposition 2.2 (*Mills' Ratio*)**.** *Let $X \sim ESN(\alpha,\tau)$ with $\alpha, \tau \in \mathbb{R}$. Then, from Proposition 2.1, as $x \to \infty$ we have*

$$\frac{1-\Phi_{\alpha,\tau}(x)}{\phi_{\alpha,\tau}(x)} \approx \begin{cases} x^{-1} & \alpha \ge 0 \\ \left\{(\alpha^2+1)x+\alpha\tau\right\}^{-1} & \alpha < 0. \end{cases}$$

Given Mills' ratio for the extended skew-normal derived in Proposition 2.2, Proposition 2.3 demonstrates sufficient conditions on the survival function $1-\Phi_{\alpha,\tau}(x)$ to conclude that the extended skew-normal distribution $\Phi_{\alpha,\tau}$ is both a Von Mises function, and is in the maximum domain of attraction of the Gumbel distribution $\Phi_{\alpha,\tau} \in \mathcal{D}(G_0)$, regardless of whether $\alpha \ge 0$ or $\alpha < 0$.

Proposition 2.3 (*Gumbel Domain of Attraction*)**.** *Let $X \sim ESN(\alpha,\tau)$ with $\alpha, \tau \in \mathbb{R}$. For $x \to \infty$, the survival function $1-\Phi_{\alpha,\tau}(x)$ can be written as*

$$1-\Phi_{\alpha,\tau}(x) = c(x)\exp\left(-\int_1^x \frac{g(v)}{f(v)}dv\right).$$

In particular, when $\alpha \ge 0$, then as $x \to \infty$

$$c(x) \to \frac{1}{\Phi(\tau/\bar\alpha)\sqrt{2\pi e}} > 0,$$
$$g(x) = 1+x^{-2} \to 1,$$
$$f(x) = x^{-1} > 0, \quad f'(x) = -x^{-2} \to 0,$$

where $\bar\alpha = (1+\alpha^2)^{1/2}$, whereas when $\alpha < 0$, without loss of generality, assume that $\alpha+\tau < 0$ and $\bar\alpha^2+\alpha\tau > 0$, then as $x \to \infty$

$$c(x) \to \frac{-\exp\left(-\frac{1+(\alpha+\tau)^2}{2}\right)}{2\pi\Phi\left(\tau/\bar\alpha\right)\left(\alpha(\alpha+\tau)+1\right)(\alpha+\tau)} > 0,$$
$$g(x) = 1+\frac{\bar\alpha^2}{\left(\bar\alpha^2 x+\alpha\tau\right)^2}+\frac{\alpha}{(\alpha x+\tau)\left(\bar\alpha^2 x+\alpha\tau\right)} \to 1,$$
$$f(x) = \frac{1}{\bar\alpha^2 x+\alpha\tau} > 0, \quad f'(x) = \frac{-\bar\alpha^2}{\left(\bar\alpha^2 x+\alpha\tau\right)^2} \to 0.$$

As a consequence, $\Phi_{\alpha,\tau}$ is both a Von Mises function and $\Phi_{\alpha,\tau} \in \mathcal{D}(G_0)$.

From Proposition 2.3 and in combination with Proposition 1.1 in Resnick (1987), it follows that the normalising constants $a_n > 0$ and $b_n \in \mathbb{R}$ in (2) can then be identified through the standard identities

$$1-\Phi_{\alpha,\tau}(b_n) = n^{-1} \quad \text{and} \quad a_n = f(b_n), \tag{5}$$

where f is given in Proposition 2.3. For practical purposes, it is usually more convenient to identify alternative normalising constants with a closed-form expression. In general terms, it is well-known that if there are normalising constants $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$, different from a_n and b_n, such that $F^n(\alpha_n x+\beta_n) \to \tilde G(x)$ converges to a non-degenerate limit $\tilde G(x)$ as $n \to \infty$, then $\tilde G$ is equal to G as given in (2) apart from some modification of the scale and location parameters (e.g. Resnick, 1987, Proposition 0.2), which does not qualitatively change the tail behaviour. In Proposition 2.4, we provide some alternative normalising constants α_n, β_n with a closed-form expression, that satisfy the conditions $\alpha_n/a_n \to 1$ and $(\beta_n - b_n)a_n \to 0$, as

$n \to \infty$ (see Leadbetter et al., 1983, Theorem 1.2.3). This therefore implies that the limiting distribution for the normalised sample maximum is still a standard Gumbel distribution.

Proposition 2.4 (*Alternative Normalising Constants*). *Let $X_1, \ldots, X_n$ be a series of iid random variables with $X_i \sim ESN(\alpha, \tau)$ for $i = 1, \ldots, n$ with $\alpha, \tau \in \mathbb{R}$. Define $M_n = \max(X_1, \ldots, X_n)$ and define the normalising constants*

$$\alpha_n = \ell_{n,0}^{-1} \quad \text{and} \quad \beta_n = \ell_{n,0} - \frac{\ln(2\sqrt{\pi}) + (1/2)\ln\ln n + \ln \Phi(\tau/\bar{\alpha})}{\ell_{n,0}} \qquad \text{if } \alpha \geq 0,$$

$$\alpha_n = \ell_{n,\alpha}^{-1} \quad \text{and} \quad \beta_n = \ell_{n,\alpha} - \frac{2\ln(2\sqrt{\pi}|\alpha|) + \ln\ln n + \ln \Phi(\tau/\bar{\alpha}) - \tau^2/2}{2\ell_{n,\alpha}} - \frac{\alpha\tau}{\bar{\alpha}^2} \quad \text{if } \alpha < 0,$$

where $\ell_{n,\alpha} := \sqrt{2\ln n(1+\alpha^2)}$ and $\bar{\alpha} = (1+\alpha^2)^{1/2}$. Then

$$\lim_{n\to\infty} \Pr\left(\frac{M_n - \beta_n}{\alpha_n} \leq x\right) = G_0(x), \quad x \in \mathbb{R}.$$

In the presence of competing normalising constants, a natural question to ask is whether the rate of convergence of $\Phi_{\alpha,\tau}^n$ to G_0 as $n \to \infty$, differs substantially when the normalising constants (α_n, β_n) in Proposition 2.4 are considered in the place of (a_n, b_n) defined by (5). Theorem 2.1 establishes the rate of convergence for each sequence of normalising constants.

Theorem 2.1 (*Convergence Rate to Gumbel Limit*). *Let $X \sim ESN(\alpha, \tau)$ with $\alpha, \tau \in \mathbb{R}$. For the normalising constants α_n, β_n defined in Proposition 2.4 we have*

$$\{\Phi_{\alpha,\tau}^n(\alpha_n x + \beta_n) - G_0(x)\} \approx \frac{G_0(x)e^{-x}}{c}\frac{(\ln\ln n)^2}{\ln n} \quad \text{as } n \to \infty,$$

where $c = 16$ when $\alpha \geq 0$ and $c = 4$ when $\alpha < 0$. For the normalising constants a_n, b_n defined in (5) we have

$$\lim_{n\to\infty} b_n^2\left[b_n^2\left\{\Phi_{\alpha,\tau}^n(a_n x + b_n) - G_0(x)\right\} - \kappa(x)G_0(x)\right] = \left(\omega(x) + \frac{\kappa^2(x)}{2}\right)G_0(x),$$

where

$$\kappa(x) = \frac{x^2 + 2x}{2}e^{-x} \quad \text{and} \quad \omega(x) = -\frac{1}{8}\left(x^4 + 4x^3 + 8x^2 + 16x\right)e^{-x}$$

when $\alpha \geq 0$, while

$$\kappa(x) = \frac{x^2 + 4x}{2(1+\alpha^2)}e^{-x} \quad \text{and} \quad \omega(x) = -\frac{\alpha^2(1+\alpha^2)^2}{8}\left\{(1+3\alpha^2)16x + \alpha^2(x^4 + 8x^3 + 24x^2)\right\}e^{-x}$$

when $\alpha < 0$.

From Theorem 2.1 it follows that when the sample maximum is normalised by (α_n, β_n) and (a_n, b_n), then the rates of convergence to the standard the Gumbel distribution are of order $(\ln\ln n)^2/\ln n$ and $1/\ln n$, respectively. That is, the rate of convergence is slower for (α_n, β_n), balancing the advantage of the closed-form expression.

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Appendix A. Proofs

A.1. Proof of Proposition 2.1

Define $x_{\alpha,\tau} := \alpha x + \tau$ for every $x, \alpha, \tau \in \mathbb{R}$. From (1) we can write

$$\frac{1 - \Phi_{\alpha,\tau}(x)}{\phi_{\alpha,\tau}(x)} = \frac{\int_x^{+\infty} \phi(t)\Phi(t_{\alpha,\tau})\mathrm{d}t}{\phi(x)\Phi(x_{\alpha,\tau})} = \frac{\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}.$$

For $x > 0$, using integration by parts gives

$$\frac{1}{x^2}\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t > \int_x^{+\infty} \frac{\Phi(t_{\alpha,\tau})}{t^2}e^{-t^2/2}\mathrm{d}t$$
$$= \frac{\Phi(x_{\alpha,\tau})}{x}e^{-x^2/2} - \int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t + \alpha\int_x^{+\infty} \frac{e^{-t^2/2}}{t}\phi(t_{\alpha,\tau})\mathrm{d}t, \quad (6)$$

from which it follows that

$$\left(1+\frac{1}{x^2}\right)\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t > \frac{\Phi(x_{\alpha,\tau})}{x}e^{-x^2/2} + \alpha\int_x^{+\infty} \frac{e^{-t^2/2}}{t}\phi(t_{\alpha,\tau})\mathrm{d}t. \quad (7)$$

We study the behaviour of the inequality (7) conditional on the sign of α.

(i) When $\alpha > 0$, by (7) we obtain

$$\left(1+\frac{1}{x^2}\right)\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t > x^{-1}\Phi(x_{\alpha,\tau})e^{-x^2/2},$$

and therefore we derive the lower bound

$$L_{\alpha,\tau}(x) = x^{-1}\left(1+\frac{1}{x^2}\right)^{-1} < \frac{\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}.$$

Since $\int_x^{+\infty} t^{-2}\Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t > 0$, then from (6) we obtain

$$\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t < \frac{\Phi(x_{\alpha,\tau})}{x}e^{-\frac{x^2}{2}}\mathrm{d}t + \alpha\int_x^{+\infty} t^{-1}e^{-\frac{t^2}{2}}\phi(t_{\alpha,\tau})\mathrm{d}t$$
$$< \frac{\Phi(x_{\alpha,\tau})}{x}e^{-\frac{x^2}{2}}\left(1+\frac{\alpha\int_x^{+\infty} e^{-\frac{t^2}{2}}\phi(t_{\alpha,\tau})\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}\right) < \frac{\Phi(x_{\alpha,\tau})}{x}e^{-\frac{x^2}{2}}\left(1+\frac{\alpha\int_x^{+\infty}\phi(t_{\alpha,\tau})\mathrm{d}t}{\Phi(x_{\alpha,\tau})}\right)$$
$$= \frac{\Phi(x_{\alpha,\tau})}{x}e^{-\frac{x^2}{2}}\left(1+\frac{1-\Phi(x_{\alpha,\tau})}{\Phi(x_{\alpha,\tau})}\right) = x^{-1}e^{-\frac{x^2}{2}}.$$

Then, we have

$$\frac{1-\Phi_{\alpha,\tau}(x)}{\phi_{\alpha,\tau}(x)} \le x^{-1}\Phi^{-1}(x_{\alpha,\tau}). \quad (8)$$

We now also need to consider the sign of $x_{\alpha,\tau}$.

(ia) When $x_{\alpha,\tau} > 0$, by (3) we have $1-\Phi(x_{\alpha,\tau}) < \phi(x_{\alpha,\tau})/x_{\alpha,\tau}$, which implies

$$\Phi(x_{\alpha,\tau})^{-1} < \left(1-\frac{\phi(x_{\alpha,\tau})}{x_{\alpha,\tau}}\right)^{-1}.$$

Then, substituting the above inequality into (8) we obtain the upper bound

$$\frac{\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} < x^{-1}\left(1-\frac{\phi(x_{\alpha,\tau})}{x_{\alpha,\tau}}\right)^{-1} = U_{\alpha,\tau}(x);$$

(ib) When $x_{\alpha,\tau} < 0$, by inequality (3) we have that $\Phi(x_{\alpha,\tau})^{-1} < -\left(x_{\alpha,\tau}^2+1\right)/\{x_{\alpha,\tau}\phi(x_{\alpha,\tau})\}$, and substituting this into (8) gives the upper bound

$$\frac{\int_x^{+\infty} \Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} < x^{-1}\left(-\frac{(x_{\alpha,\tau})^2+1}{x_{\alpha,\tau}\phi(x_{\alpha,\tau})}\right) = U_{\alpha,\tau}(x).$$

(ii) When $\alpha < 0$, using the property

$$\phi\left(\frac{x-\mu_1}{\sigma_1}\right)\phi\left(\frac{x-\mu_2}{\sigma_2}\right) = \phi\left(\frac{x\sqrt{\sigma_1^2+\sigma_2^2}}{\sigma_1\sigma_2} - \frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sqrt{\sigma_1^2+\sigma_2^2}\sigma_1\sigma_2}\right)\phi\left(\frac{\mu_1-\mu_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right),$$

we can write

$$\exp\left(-\frac{(t_{\alpha,\tau})^2+t^2}{2}\right) = 2\pi\phi\left(\frac{t+\tau\alpha^{-1}}{\alpha^{-1}}\right)\phi(t) = 2\pi\phi\left(\bar{\alpha}t+\frac{\tau\alpha^{-1}}{\bar{\alpha}}\right)\phi\left(\frac{\tau}{\bar{\alpha}}\right),$$

where $\bar{\alpha}=\sqrt{1+\alpha^2}$, which then gives

$$\begin{aligned}\alpha\int_x^{+\infty} e^{-t^2/2}\phi(t_{\alpha,\tau})\mathrm{d}t &= \alpha\sqrt{2\pi}\int_x^{+\infty}\phi\left(t\bar{\alpha}+\frac{\tau\alpha^{-1}}{\bar{\alpha}}\right)\phi\left(\frac{\tau}{\bar{\alpha}}\right)\mathrm{d}t\\ &= \frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)\right\}.\end{aligned} \tag{9}$$

Hence, starting from (7), a lower bound is of the form

$$\begin{aligned}\left(1+\frac{1}{x^2}\right)\int_x^{+\infty}\Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t &> \frac{\Phi(x_{\alpha,\tau})}{x}e^{-x^2/2}\left(1+\frac{\alpha\int_x^{+\infty}t^{-1}e^{-t^2/2}\phi(t_{\alpha,\tau})\mathrm{d}t}{x^{-1}\Phi(x_{\alpha,\tau})e^{-x^2/2}}\right)\\ &> \frac{\Phi(x_{\alpha,\tau})}{x}e^{-x^2/2}\left(1+\frac{\alpha\int_x^{+\infty}e^{-t^2/2}\phi(t_{\alpha,\tau})\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}\right)\\ &= \frac{\Phi(x_{\alpha,\tau})}{x}e^{-x^2/2}\left(1+\frac{\frac{\alpha}{\bar{\alpha}}e^{-\frac{\tau^2}{2\bar{\alpha}^2}}\left\{1-\Phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}\right).\end{aligned} \tag{10}$$

Similarly, an upper bound is given by

$$x\int_x^{+\infty}\Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t < \int_x^{+\infty}t\Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t = \Phi(x_{\alpha,\tau})e^{-x^2/2}+\int_x^{+\infty}\alpha\phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t,$$

and hence, using (9) we obtain

$$\begin{aligned}\int_x^{+\infty}\Phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t &< x^{-1}\Phi(x_{\alpha,\tau})e^{-x^2/2}\left(1+\frac{\int_x^{+\infty}\alpha\phi(t_{\alpha,\tau})e^{-t^2/2}\mathrm{d}t}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}\right)\\ &= x^{-1}\Phi(x_{\alpha,\tau})e^{-x^2/2}\left(1+\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}}\right).\end{aligned} \tag{11}$$

As before, we also need to consider the sign of $x_{\alpha,\tau}$.

(ii*a*) When $x_{\alpha,\tau}>0$ and $x+\alpha x_{\alpha,\tau}>0$ then we have $x\bar{\alpha}+(\alpha\tau)/\bar{\alpha}>0$. Therefore, by (3) we obtain $1-\Phi(x_{\alpha,\tau})<\phi(x_{\alpha,\tau})/x_{\alpha,\tau}$, and using the equality

$$e^{-\tau^2/(2\bar{\alpha}^2)}\phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right) = e^{-x^2/2}\phi\left(x_{\alpha,\tau}\right) \tag{12}$$

we have that

$$\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} > \frac{\alpha\left(\bar{\alpha}^2x+\alpha\tau\right)^{-1}}{1/\phi(x_{\alpha,\tau})-1/x_{\alpha,\tau}}.$$

Thus, substituting the above equality into (10) leads to the desired lower bound $L_{\alpha,\tau}(x)$. Further, from (3) we also have $\Phi(x_{\alpha,\tau})<1-\phi(x_{\alpha,\tau})x_{\alpha,\tau}/(x_{\alpha,\tau}^2+1)$, which, combined with (12), gives

$$\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} < \frac{\frac{\alpha(\bar{\alpha}^2x+\alpha\tau)}{(\bar{\alpha}^2x+\alpha\tau)^2+\bar{\alpha}^2}}{\phi(x_{\alpha,\tau})^{-1}-\frac{x_{\alpha,\tau}}{x_{\alpha,\tau}^2+1}}.$$

Applying the above inequality to (11) leads to the upper bound $U_{\alpha,\tau}(x)$.

(ii*b*) When $x_{\alpha,\tau}>0$ and $x+\alpha x_{\alpha,\tau}<0$ then we have $\bar{\alpha}x+(\alpha\tau)/\bar{\alpha}<0$. Therefore, by (3) we obtain

$$1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right) < 1+\frac{\bar{\alpha}}{\bar{\alpha}^2x+\alpha\tau}\frac{\phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)}{1+1/(\bar{\alpha}x+\alpha\tau/\bar{\alpha})^2},$$

which, combined with (12), gives

$$\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha})}\left\{1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} > \frac{\frac{\alpha}{\bar{\alpha}}\phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)^{-1}+\frac{\alpha(\bar{\alpha}^2x+\alpha\tau)}{(\bar{\alpha}^2x+\alpha\tau)^2+\bar{\alpha}^2}}{1/\phi(x_{\alpha,\tau})-1/x_{\alpha,\tau}}.$$

Together with (10) the above inequality leads to the desired lower bound $L_{\alpha,\tau}(x)$. By (3) we also have

$$1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right) > 1+\frac{\phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)}{\bar{\alpha}x+\alpha\tau/\bar{\alpha}},$$

and from this it follows that

$$\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} < \frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left(1+\frac{\phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)}{\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}}\right)}{e^{-x^2/2}\left\{1-x_{\alpha,\tau}\phi(x_{\alpha,\tau})/(x^2_{\alpha,\tau}+1)\right\}}.$$

Together with (11) the above inequality leads to the upper bound $U_{\alpha,\tau}(x)$.

(iic) When $x_{\alpha,\tau}<0$, since $\alpha/\bar{\alpha}<0$ and $\bar{\alpha}x+(\alpha\tau)\bar{\alpha}>0$, then by (3) we have

$$1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right) < \frac{\phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)}{\bar{\alpha}x+\alpha\tau/\bar{\alpha}},$$

and thus

$$\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)\right\} > \frac{\alpha}{\bar{\alpha}\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right)}e^{-\tau^2/(2\bar{\alpha}^2)}\phi\left(\bar{\alpha}x+\frac{\alpha\tau}{\bar{\alpha}}\right). \tag{13}$$

Furthermore, by (3) we also have

$$\Phi(x_{\alpha,\tau})e^{-x^2/2} > -\frac{x_{\alpha,\tau}}{x^2_{\alpha,\tau}+1}\phi(x_{\alpha,\tau})e^{-x^2/2},$$

and combined with (12), (13) then becomes

$$\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} > -\frac{\alpha(x^2_{\alpha,\tau}+1)}{(\bar{\alpha}^2x+\alpha\tau)x_{\alpha,\tau}}.$$

Together with (10) the above inequality provides the lower bound $L_{\alpha,\tau}(x)$. Again applying (3) we have $\Phi(x_{\alpha,\tau})<\phi(x_{\alpha,\tau})/x_{\alpha,\tau}$ and

$$1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right) > \frac{\phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)}{\bar{\alpha}x+\alpha\tau/\bar{\alpha}}\left(1+(\bar{\alpha}x+\alpha\tau/\bar{\alpha})^{-2}\right)^{-1}.$$

Using $\alpha/\bar{\alpha}<0$ and (3) gives

$$\frac{\frac{\alpha}{\bar{\alpha}}e^{-\tau^2/(2\bar{\alpha}^2)}\left\{1-\Phi\left(\bar{\alpha}x+\alpha\tau/\bar{\alpha}\right)\right\}}{\Phi(x_{\alpha,\tau})e^{-x^2/2}} < -\frac{\alpha x_{\alpha,\tau}(x+\alpha x_{\alpha,\tau})}{(\bar{\alpha}^2x+\alpha\tau)^2+\alpha^2+1},$$

which together with (10) provides the upper bound $U_{\alpha,\tau}(x)$.

A.2. Proof of Proposition 2.2

Let $x_{\alpha,\tau}:=\alpha x+\tau$ and $\bar{\alpha}=(1+\alpha^2)^{1/2}$. First, note that when $\alpha>0$, then $x_{\alpha,\tau}$ becomes positive when $x\to\infty$, regardless of the value of τ. From Proposition 2.1 and noting that

$$\lim_{x\to\infty}\left(1-\frac{\phi(x_{\alpha,\tau})}{x_{\alpha,\tau}}\right)^{-1}=1$$

we can obtain

$$L_{\alpha,\tau}(x)=\frac{1}{x+x^{-1}}\approx\frac{1}{x},\quad U_{\alpha,\tau}(x)\approx\frac{1}{x}.$$

Conversely, when $\alpha<0$, then $x_{\alpha,\tau}$ becomes negative when $x\to\infty$, regardless of the value of τ. From Proposition 2.1 we have that the dominating term for the lower and upper bounds is

$$x^{-1}\left(1-\frac{\alpha x_{\alpha,\tau}}{\bar{\alpha}^2x+\alpha\tau}\right).$$

As a consequence,

$$L_{\alpha,\tau}(x)\approx\frac{1}{\bar{\alpha}^2x+\alpha\tau},\quad U_{\alpha,\tau}(x)\approx\frac{1}{\bar{\alpha}^2x+\alpha\tau}.$$

A.3. Proof of Proposition 2.3

Define $x_{\alpha,\tau} := \alpha x + \tau$ and $\bar{\alpha} = (1+\alpha^2)^{1/2}$ for every $x, \alpha, \tau \in \mathbb{R}$. When $\alpha \geq 0$, by Proposition 2.2 as $x \to \infty$ we have

$$1 - \Phi_{\alpha,\tau}(x) \approx \frac{\Phi(x_{\alpha,\tau})}{\Phi(\tau/\bar{\alpha})} \frac{e^{-x^2/2}}{x\sqrt{2\pi}} = \frac{\Phi(x_{\alpha,\tau})}{\Phi(\tau/\bar{\alpha})} \frac{1}{\sqrt{2\pi e}} e^{\left(\frac{1}{2} - \ln x - \frac{x^2}{2}\right)}$$
$$= \frac{\Phi(x_{\alpha,\tau})}{\Phi(\tau/\bar{\alpha})} \frac{1}{\sqrt{2\pi e}} \exp\left(-\int_1^x \frac{\frac{t^2+1}{t^2}}{\frac{1}{t}} \mathrm{d}t\right)$$
$$= c(x) \exp\left(-\int_1^x \frac{g(t)}{f(t)} \mathrm{d}t\right).$$

It follows that

$$\lim_{x\to\infty} c(x) = \frac{1}{\Phi(\tau/\bar{\alpha})\sqrt{2\pi e}}, \quad \lim_{x\to\infty} g(x) = 1, \quad \lim_{x\to\infty} f'(x) = 0.$$

When $\alpha < 0$, from Proposition 2.2 as $x \to \infty$ we have

$$1 - \Phi_{\alpha,\tau}(x) \approx \frac{-\phi(x)\phi(x_{\alpha,\tau})}{\left(\bar{\alpha}^2 x + \alpha\tau\right) x_{\alpha,\tau} \Phi(\tau/\bar{\alpha})}$$
$$= \frac{e^{-\frac{1+(\alpha+\tau)^2}{2}}}{2\pi\Phi(\tau/\bar{\alpha})} \exp\left\{-\frac{1}{2}(x^2-1)(\alpha^2+1) - \alpha\tau(x-1) - \ln(\alpha(x_{\alpha,\tau})+x) - \ln(-x_{\alpha,\tau})\right\}$$
$$= \frac{e^{-\frac{1+(\alpha+\tau)^2}{2}}}{2\pi\Phi(\tau/\bar{\alpha})} \exp\left\{-\ln(\alpha(\alpha+\tau)+1) - \ln(-(\alpha+\tau)) - \int_1^x \left(\bar{\alpha}^2 t + \lambda\tau + \frac{\bar{\alpha}^2}{\alpha t_{\alpha,\tau}+t} + \frac{\alpha}{t_{\alpha,\tau}}\right) \mathrm{d}t\right\}$$
$$= \frac{-e^{-\frac{1+(\alpha+\tau)^2}{2}}(\alpha+\tau)^{-1}}{2\pi\Phi(\tau/\bar{\alpha})\{\alpha(\alpha+\tau)+1\}} \exp\left\{-\int_1^x \frac{1 + \frac{\alpha^2+1}{(\bar{\alpha}^2 t+\alpha\tau)^2} + \frac{\alpha}{(\alpha t+\tau)(\bar{\alpha}^2 t+\alpha\tau)}}{\frac{1}{\bar{\alpha}^2 t+\alpha\tau}} \mathrm{d}t\right\}$$
$$= c(x) \exp\left(-\int_1^x \frac{g(t)}{f(t)} \mathrm{d}t\right),$$

assuming $\bar{\alpha} + \alpha\tau > 0$ and $\alpha + \tau < 0$. It follows that

$$\lim_{x\to\infty} c(x) = \frac{-e^{-\frac{1+(\alpha+\tau)^2}{2}}(\alpha+\tau)^{-1}}{2\pi\Phi(\tau/\bar{\alpha})\{\alpha(\alpha+\tau)+1\}} > 0, \quad \lim_{x\to\infty} g(x) = 1, \quad \lim_{x\to\infty} f'(x) = 0.$$

Therefore, by Proposition 1.1(a) and Corollary 1.7 in Resnick (1987) we have that $\Phi_{\alpha,\tau}$ is a Von Mises function and $\Phi_{\alpha,\tau} \in \mathcal{D}(G_0)$.

A.4. Proof of proposition 3.4

Recall that for brevity we write $\ell_{n,\alpha} = \sqrt{2(1+\alpha^2)\ln n}$ and $\bar{\alpha} = (1+\alpha^2)^{1/2}$ for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. By Proposition 2.3 we know that $\Phi_{\alpha,\tau} \in \mathcal{D}(G_0)$ and therefore by Proposition 1.1 in Resnick (1987) we have that the normalising constants $a_n > 0$ and $b_n \in \mathbb{R}$ can be obtained by solving the equation in (5). Here we derive some approximations for a_n and b_n. We distinguish two cases. First, we consider $\alpha \geq 0$. By Proposition 2.3 the second equation of (5) gives $a_n = 1/b_n$ and by Proposition 2.2 the left hand-side term of the first equation can be approximated as $1 - \Phi_{\alpha,\tau}(b_n) \approx \phi_{\alpha,\tau}(b_n)/b_n$ as $n \to \infty$, and so through tail equivalence we can focus on the equation $n\phi_{\alpha,\tau}(b_n) = b_n$. Taking the logarithm on both sides we obtain

$$\ln n - \ln b_n - \frac{1}{2}\ln 2\pi - \frac{b_n^2}{2} + \ln\Phi(\alpha b_n + \tau) - \ln\Phi(\tau/\bar{\alpha}) = 0. \tag{14}$$

Dividing (14) by b_n^2 gives $b_n = \ell_{n,0} + o(1)$ and

$$\log b_n = \frac{1}{2}(\ln 2 + \ln\ln n) + o(1). \tag{15}$$

We set $\alpha_n = 1/\ell_{n,0}$. Using the fact that $\Phi(\alpha b_n + \tau) \to 1$ as $n \to \infty$ and plugging (15) in (14) we obtain

$$b_n = \ell_{n,0} - \frac{1/2\ln\ln n + \ln(2\sqrt{\pi}) + \ln 2\Phi(\tau/\bar{\alpha})}{\ell_{n,0}} + o\left(1/\ell_{n,0}\right) = \beta_n + o(\alpha_n).$$

In the second case we assume $\alpha < 0$. By Proposition 2.2 the left hand-side term of the first equation can be approximated by $\phi_{\alpha,\tau}(b_n)/\{(1+\alpha^2)b_n+\alpha\tau\}$ as $n \to \infty$, and so through tail equivalence we can focus on the equation $n\phi_{\alpha,\tau}(b_n) = (1+\alpha^2)b_n+\alpha\tau$. By taking the logarithm on both sides and noting that from Proposition 2.1

$$\ln \Phi(\alpha b_n+\tau) \approx -\frac{1}{2}\ln 2\pi - \frac{(\alpha b_n+\tau)^2}{2} - \ln(-(\alpha b_n+\tau)), \quad n \to \infty,$$

then we obtain

$$\ln n - \ln\{(1+\alpha^2)b_n+\alpha\tau\} - \ln 2\pi\alpha\tau b_n - \frac{\tau^2}{2} - \frac{b_n^2\bar{\alpha}^2}{2} + \ln \Phi\{-(\alpha b_n+\tau)\} - \ln \Phi(\tau/\bar{\alpha}) = 0. \qquad (16)$$

Dividing (16) by b_n^2 then gives $b_n = \ell_{n,\alpha} + o(1)$. We set $\alpha_n = 1/\ell_{n,\alpha}$. Plugging b_n in (16) we obtain

$$b_n = \ell_{n,\alpha} - \frac{2\ln(2\sqrt{\pi}|\alpha|) + \ln\ln n + \ln \Phi\,(\tau/\bar{\alpha}) - \tau^2/2}{2\ell_{n,\alpha}} - \frac{\alpha\tau}{1+\alpha^2} + o(1/\ell_{n,\alpha}) = \beta_n + o(\alpha_n).$$

Finally, as in both cases $\alpha \geq 0$ and $\alpha < 0$ we have $a_n/\alpha_n \to 1$ and $(\alpha_n - \beta_n)/a_n \to 0$ as $n \to \infty$, then by Resnick (1987, Proposition 0.2) we have $\Phi_{\alpha,\tau}(\alpha_n x + \beta_n) \to G_0(x)$ as $n \to \infty$.

A.5. Proof of Theorem 2.1

Let $u_n = \alpha_n x + \beta_n$ and $v_n = \alpha_n x + \beta_n$ where α_n and β_n are respectively defined as in Appendix A.4 when $\alpha \geq 0$ and $\alpha < 0$. Let $\bar{\alpha} = (1+\alpha^2)^{1/2}$ for any $\alpha \in \mathbb{R}$. When $\alpha \geq 0$ it is easy to check that $u_n^{-2} \sim \ell_{n,0}^{-2}$ as $n \to \infty$, which implies that $O(u_n^{-2}) = O((\ln n)^{-1})$, and

$$u_n^{-1} = \ell_{n,0}^{-1}\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\}.$$

Furthermore, by Proposition 2.1(ia) we can write

$$1 - \Phi_{\alpha,\tau}(u_n) = u_n^{-1}\phi_{\alpha,\tau}(u_n)\{1 + O(u_n^{-2})\}.$$

Defining $\vartheta_n = n\{1 - \Phi(u_n;\alpha,\tau)\}$, we may then write

$$\vartheta_n = e^{-x}\left[1 - \frac{(\ln\ln n)^2}{16\ln n}\{1+o(1)\}\right].$$

Setting $\vartheta = e^{-x}$ then gives

$$\vartheta - \vartheta_n = e^{-x}\frac{(\ln\ln n)^2}{16\ln n}\{1+o(1)\}.$$

In the case when $\alpha < 0$ it is also easy to check that

$$v_n^{-2} = \ell_{n,0}^{-2}\bar{\alpha}^2\left[1 + O\left\{\left(\frac{\ln\ln n}{\ln n}\right)^2\right\}\right]$$

which implies that $O(v_n^{-2}) = O((\ln n)^{-1})$, and

$$v_n^{-1} = \ell_{n,0}^{-1}\bar{\alpha}\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\}.$$

Furthermore, by Proposition 2.1(iic) we have

$$1 - \Phi_{\alpha,\tau}(v_n) = -\phi(x)\phi(v_{\alpha,\tau})\left\{(\bar{\alpha}v_n + \alpha\tau)\, v_{n,\alpha,\tau}\Phi(\tau/\bar{\alpha})\right\}^{-1}\{1 + O(v_n^{-2})\},$$

where $v_{n,\alpha,\tau} = \alpha v_n + \tau$. Thus, when $\vartheta_n = n(1 - \Phi(v_n;\alpha,\tau))$, using the additional approximation $v_n = \ell_{n,\alpha} + o(1)$, we can write

$$\vartheta_n = e^{-x}\left[1 - \frac{(\ln\ln n)^2}{4\ln n}\{1+o(1)\}\right],$$

from which we obtain

$$\vartheta - \vartheta_n = e^{-x}\frac{(\ln\ln n)^2}{4\ln n}\{1+o(1)\}.$$

Then apply Leadbetter et al. (1983, Theorem 2.4.2) to complete the proof of the first assertion of the theorem.

Focusing on the normalising constants a_n and b_n given in (5), we require the following lemma to determine the speed of convergence to the Gumbel distribution.

Lemma 1. *Let $h_{\alpha,\tau}(x;b_n) = n\log\Phi_{\alpha,\tau}(f(b_n)x+b_n)+e^{-x}$, where the normalising constant b_n is given by (5) and f is the auxiliary function defined in Proposition 2.3. Then*

$$\lim_{n\to\infty} b_n^2\left(b_n^2 h_{\alpha,\beta}(x;b_n)-\kappa(x)\right)=\omega(x),$$

where $\kappa(x)$ and $\omega(x)$ depend on the sign of the slant parameter α and are defined in the statement of Theorem 2.1.

The proof of Lemma 1 is provided in the Supplementary Material. Lemma 1 indicates that $h_{\alpha,\tau}(x,b_n)\to 0$ as n gets large and

$$\left|\sum_{i=3}^{\infty}\frac{h_{\alpha,\tau}^{i-3}(x,b_n)}{i!}\right| < \exp\left(h_{\alpha,\tau}(x,b_n)\right)\to 1,\quad n\to\infty.$$

Then, noting that $\exp\left\{h_{\alpha,\tau}(x,b_n)\right\}=\Phi_{\alpha,\tau}^n(a_nx+b_n)G_0(x)^{-1}$ and applying Lemma 1 once more, we have

$$\begin{aligned}
b_n^2\left[b_n^2\left(\Phi_{\alpha,\tau}^n(a_nx+b_n)-G_0(x)\right)-\kappa(x)G_0(x)\right] &= b_n^2\left[b_n^2\left(\frac{\Phi_{\alpha,\tau}^n(a_nx+b_n)}{G_0(x)}-1\right)-\kappa(x)\right]G_0(x)\\
&= b_n^2\left[b_n^2\left(\exp\left\{h_{\alpha,\tau}(x,b_n)\right\}-1\right)-\kappa(x)\right]G_0(x)\\
&= b_n^2\left[b_n^2\left(h_{\alpha,\tau}(x,b_n)+\frac{h_{\alpha,\tau}(x,b_n)^2}{2}+\sum_{i=3}^{\infty}\frac{h_{\alpha,\tau}(x,b_n)^i}{i!}\right)-\kappa(x)\right]G_0(x)\\
&= \left[b_n^2\left(b_n^2h_{\alpha,\tau}(x,b_n)-\kappa(x)\right)+b_n^4h_{\alpha,\tau}(x,b_n)^2\left(\frac{1}{2}+h(x,b_n;\alpha,\tau)\sum_{i=3}^{\infty}\frac{h_{\alpha,\tau}(x,b_n)^{i-3}}{i!}\right)\right]G_0(x)\\
&\to\left[\omega(x)+\frac{\kappa^2(x)}{2}\right]G_0(x),\quad n\to\infty.
\end{aligned}$$

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2018.09.018.

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