On some features of the skewed families of max-stable processes

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Motivation

- What is the maximum value a variable is expected to reach across a region within the next 10 years? $\Longrightarrow$ SPATIAL EXTREMES
- Environmental extremes are spatial and often highly skewed

Figure: Tornado in Oklahoma, May 1999

Outline

- Max-stable processes
- Extremal dependence models
- Sampling from max-stables processes

Max-stable processes (1)

- $X_1, X_2, \ldots,$ be i.i.d replicates of $X(s), s \in \mathcal{S} \subset \mathbb{R}^d$,

$$\left\{ \max_{i=1,\ldots,n} \frac{X_i(s) - b_n(s)}{a_n(s)} \right\}_{s \in \mathcal{S}} \xrightarrow{d} \{Y(s)\}_{s \in \mathcal{S}} \tag{1}$$

for some continuous functions $a_n(s) > 0$ and $b_n(s)$.

- $Y_0(s)$ be the limiting process with unit Fréchet margins

$$P\left\{Y_0(s_j) \le y(s_j), j \in I\right\} = \exp\left\{-V_0(y(s_j), j \in I)\right\}$$

where

$$V_0\{y(s_j), j \in I\} = d \int_{\mathcal{W}_d} \max_{j \in I} \left(\frac{w_j}{y(s_j)} \right) \mathrm{d}H(\boldsymbol{w}).$$

Max-stable processes (2)

Theorem (Spectral representation) (e.g. Schlather, 2002)
Let $\{R_i\}_{i \geq 1}$ be the points of a Poisson process on $\mathbb{R}^+$ with intensity $\xi r^{-(\xi+1)}$, $\xi > 0$.
Let $X^+ = \max_s(0, X(s))$, $\mu^+(s) = \mathbb{E}[\{X^+(s)\}^\xi] < \infty$ and X_i^+, $i = 1, 2, \ldots$ be i.i.d copies of X^+. Then

$$Y(s) = \max_{i=1,2,\ldots} \{R_i X_i^+(s)\} / \{\mu^+(s)\}^{1/\xi}, \quad s \in \mathcal{S}, \tag{2}$$

is a max-stable process with ξ-Fréchet 1-d distributions.

The exponent function of (2) is

$$V\{y(s_j), j \in I\} = \mathbb{E}\left[\max_{j \in I} \left\{ \frac{X^+(s_j)^\xi}{\mu^+(s_j) y(s_j)^\xi} \right\}\right].$$

Existing extremal dependence models

The most widely used models are:

1. Smith model (Smith, 1990) U_i point of PP on $\mathbb{R}^d$ with positive intensity measure $\mu(\mathrm{d}u)$, $X_i(s) = \phi_d(s - U_i; 0, \Sigma), s \in \mathcal{S}$
2. Schlather model (Schlather, 2002) $\xi = 1$ and $X_i(s)$ are i.i.d. copies of a weakly stationary GP with isotropic correlation function $\rho(h)$
3. Brown-Resnick model (Kabluchko et al., 2009) $X_i(s) = \exp\left\{\epsilon_i(s) - \sigma^2(s)/2\right\}$ where ϵ is a GP with stationary increments and $\sigma^2(s) = \mathrm{Var}\{\epsilon(s)\}$;
4. Extremal-t (Opitz, 2013) Schlather model but $\xi > 0$;

Drawbacks: None can deal with skewness, all stationary.
Question: What type of extreme dependence structures do we get working with skew families?

New extremal dependence models (1)

Proposition: (Extremal Skew-t)(B., Padoan & Sisson, 2016)
Consider the spect. rep. (2) where $X(s)$ is a skew-normal process, then $Y(s)$ is a max-stable process with exponent function

$$V\{y(s_j), j \in I\} = \sum_{j=1}^{d} \frac{1}{y(s_j)^{\xi}} \Psi_{d-1}\left[\{q_i, i \in I_j\}^{\top}; \bar{\Sigma}_j, \alpha_j^*, \tau_j^*, \nu + 1\right],$$

where Ψ_{d-1} is a $d-1$-dimensional extended skew-t cdf.

1. $X(s)$ is a not strictly stationary
2. $Y(s)$ is a non-stationary max-stable process.
3. $\alpha(s) = 0$, $\forall s \in S$, $\Longrightarrow$ Extremal-t;
4. $\alpha(s) = 0$ and $\nu = 1 \Longrightarrow$ Schlather;

New extremal dependence models (2)

Proposition: (Skew-Smith)(B., Dombry & Padoan, 2017)
Consider the spect. rep. (2) where

$$X_i(s) = \phi_2(s - U_i; 0, \Sigma, \boldsymbol{\alpha}), \quad s \in \mathcal{S} = \mathbb{R}^2,$$

then $Y(s)$ is a max-stable process with exponent function

$$V\{y(s_1), y(s_2)\} = \frac{\Phi(d;\sigma,\alpha^*)}{y(s_1)} + \frac{\Phi(h_1 - d - d_1 h_2;\sigma,-\alpha^*)}{y(s_2)},$$

where $d \in \mathbb{R}$ is a function of $y(s_1), y(s_2), \mathbf{h}, \sigma$ and $\boldsymbol{\alpha}$,
$d_1 \in \mathbb{R}$ is a function of $\mathbf{h}, \sigma$ and $\boldsymbol{\alpha}$.

Proposition: (Extremal Skew-Normal)(B., Dombry & Padoan, 2017)
Consider a skew-Normal process with appropriate $a_n(s)$ and $b_n(s)$, then, using (1), $Y(s)$ is a max-stable process with exponent function

$$V\{x(s_j), j \in I\} = \sum_{j=1}^{d} \frac{1}{x(s_j)} \Phi_{d-1}\left[\{q_i, i \in I_j\}^\top; \bar{\Delta}_j, \alpha_j^*, \tau_j^*, \tau_j\right],$$

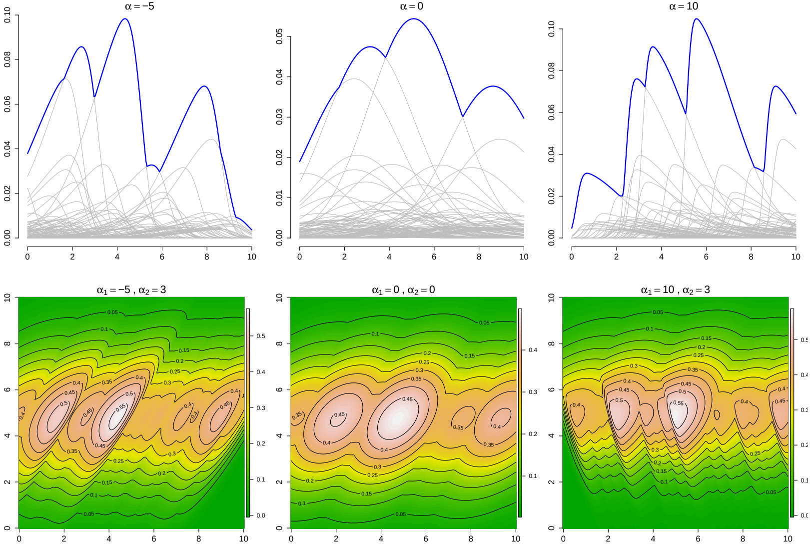
where Φ_{d-1} is a $d-1$-dimensional extended skew-Normal cdf.

Unconditional simulations (1)

- Naive approach based on the limiting result (1):
  - Find the rescaling functions a_n and b_n;
  - Compute the maxima for n large enough;
  - Drawbacks: slow convergence, requires VERY large n.

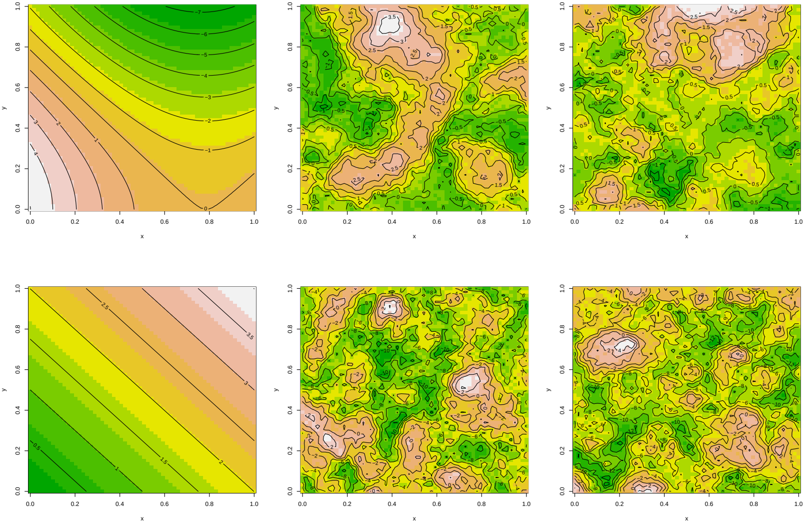
- Second approach using the spec. rep. result (2):
  - Assume $\{R_i, i \in \mathbb{N}\}$ sorted in decreasing order;
  - $X(s)$ is uniformly bounded $\Longrightarrow \exists$ a finite number of elements that contribute to the componentwise maxima.
  - Need to introduce random translations if X is not stationary (e.g. Brown Resnick).

Unconditional simulations (2)



Simulation of the univariate and bivariate Skew-Smith model.

Unconditional simulations (3)



Bivariate Extremal skew-t (top) and skew-normal (bottom) model.

Conditional simulations (1)

Prediction problem: *Obs to predict process behaviour at unobs locations?*

$\Longrightarrow$ **Goal:** Sample from $Y(s)|\{Y(s_1) = y_1, \ldots, Y(s_k) = y_k\}$

- Define $\Phi = \{\phi_i, i \in \mathbb{N}\}$ the PPP on $\mathbb{C}^+(\mathcal{S})$, with $\phi_i(s) = R_i X_i(s)$ and intensity measure

$$\Lambda_{\mathbf{s}}(A) = \int_0^\infty \mathbb{P}\{rY(\mathbf{s}) \in A\} r^{-2} \mathrm{d}r = \int_A \lambda_{\mathbf{s}}(\mathrm{v}) \mathrm{dv},$$

for all Borel sets $A \subset \mathbb{R}^d$.

- Φ can be decomposed into 2 sets of functions:
  - Extremal: $\Phi^+ = \bigcup_{j=1}^k \{\phi \in \Phi : \phi(s_j) = y_j\}$;
  - Sub-extremal $\Phi^- = \{\phi \in \Phi : \phi(s_j) < y_j; j = 1, \ldots, k\}$;
  - $\Phi^+ = \{\phi_1^+, \ldots, \phi_l^+; l = 1, \ldots, k\}$;
  - $\Phi^- = \Phi \setminus \Phi^+$;
  - Denote by $\tau = (\tau_1, \ldots, \tau_l) \in \mathcal{P}_k$ a partition of $\{x_1, \ldots, x_k\}$;

Conditional simulations (2)

Dombry et al. (2013) introduce a three-step procedure

Step 1. Sample a random partition $p \in \mathcal{P}_k$ from

$$\mathbb{P}\{p = \tau | Y(\mathbf{s}) = \boldsymbol{y}\}, \quad \text{(requires } \lambda_{\mathbf{s}})$$

Step 2. Given $p = \tau$, sample $\phi_1^+(s), \dots, \phi_l^+(s)$ independently from

$$\mathbb{P}\{\phi_j^+ \in \mathsf{d}f | Y(\mathbf{s}) = \boldsymbol{y}; p = \tau\}, \quad \text{(requires } \lambda_{\mathsf{t}|\mathbf{s},\mathsf{v}})$$

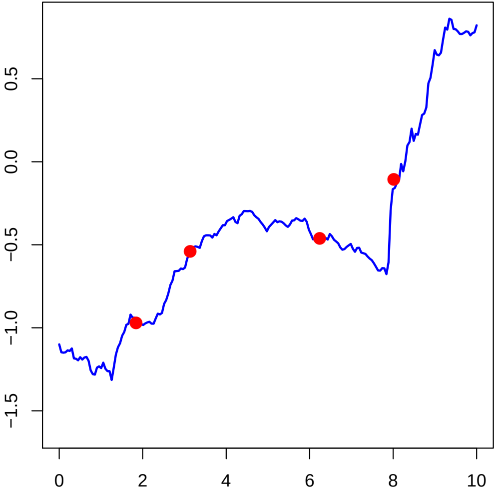
and define $Y^+ = \max_j \phi_j^+(s)$.

Step 3. Sample a PPP $\{R_i\}_{i \geq 1}$ and $\{X_i\}_{i \geq 1}$ as in (2) and define

$$Y^- = \max_{i \geq 1} R_i X_i(s) \mathbb{I}_{\{R_i X_i(s) < y\}}.$$

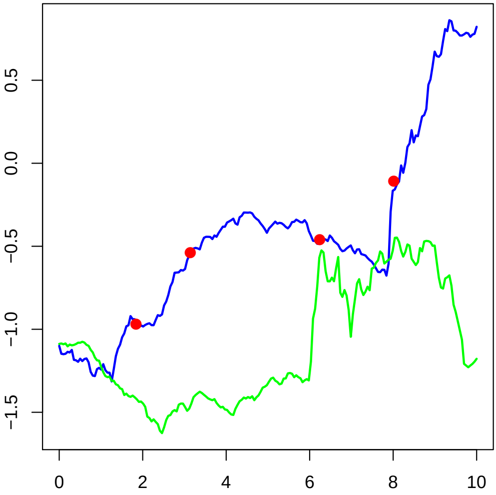
$\Longrightarrow \max(Y^+, Y^-)$ follows cond. dist of $Y(s) | \{Y(\mathbf{s}) = \boldsymbol{y}\}$.

Conditional simulations (3)



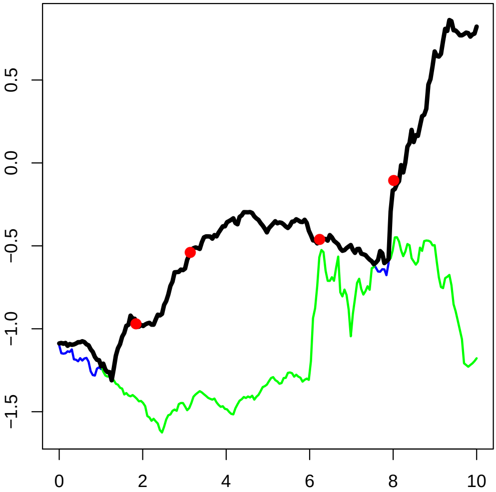
Extremal function Φ^+.

Conditional simulations (3)



Sub-extremal function Φ^{-}.

Conditional simulations (3)



Conditional max-stable process $\max(\Phi^+, \Phi^-)$.

Conditional simulations for the Extremal Skew-t

Proposition: (Beranger et al., 2017)

The intensity function of the Extremal Skew-t model is given by

$$\lambda_{\mathbf{s}}(\mathsf{v}) = \frac{2^{(\nu-2)/2}\nu^{-k+1}\Gamma\left(\frac{k+\nu}{2}\right)\Psi\left(\tilde{\alpha}_{\mathbf{s}}\sqrt{k+\nu};k+\nu\right)\prod_{i=1}^{k}\left(m_i^+ v_i^{\nu-1}\right)^{1/\nu}}{\pi^{k/2}|\bar{\Omega}_{\mathbf{s}}|^{1/2}\left(\mathsf{v}_*^\top\bar{\Omega}_{\mathbf{s}}^{-1}\mathsf{v}_*\right)^{(k+\nu)/2}\Phi(\tau_{\mathbf{s}}\{1+Q_{\bar{\Omega}_{\mathbf{s}}}(\alpha_{\mathbf{s}})\}^{-1/2})}.$$

- The conditional intensity function is also available its expression is a little messy...
- If $\alpha_{\mathbf{s}} = 0 \Longrightarrow$ Extremal-t (Ribatet, 2013).

Good news!!! The model is complex but (conditional) simulation is feasible.

Brief Conclusion

Main results:

- Various new skewed models: skew-Smith, extremal skew-Normal, extremal Skew-t;
- Conditional simulation for the extremal skew-t.

Future works:

- Extension to Skew-elliptical models;
- Link between models;
- Conditional simulation for skew-Smith model?

Many thanks for your attention!