First steps in the analysis of Symbolic Data

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Outline

1. What is Symbolic Data Analysis?
2. Existing and new SDA models
3. An example in EVT
4. Discussion
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What is Symbolic Data Analysis?

Rise of non-standard data forms

Standard statistical methods analyse classical datasets

E.g. \( x_1, \ldots, x_n \) where \( x_i \in \mathcal{X} = \mathbb{R}^p \)

However: Increasingly see non-standard data forms for analysis.

Simple non-standard forms:

- Can arise as result of measurement process
- Blood pressure naturally recorded as (low, high) interval
- Particulate matter directly recorded as counts within particle diameter ranges i.e. histogram
What is Symbolic Data Analysis?

Example: Discretised data = histogram

- E.g. point (4.0, 0.0) actually lies within [3.95, 4.05] × [−0.05, 0.05)
- Strong discretisation could have undesired inferential impact
Symbolic Data Analysis

- Established by Diday & coauthors in 1990s.
- Basic unit of data is a distribution rather than usual datapoint.
  - interval \((a, b)\)
  - \(p\)-dim hyper-rectangle
  - histogram
  - weighted list etc.
  - can be complicated by “rules”
- Classical data are a special case of symbolic data:
  - E.g. symbolic interval \(s = (a, b)\) equivalent to classical data point \(x\) if \(x = a = b\).
  - Or histogram \(\rightarrow \{x_i\}\) as \# bins \(\rightarrow \infty\).

\[ \implies \text{symbolic analyses must reduce to classical methods.} \]
What is Symbolic Data Analysis?

How do symbolic data arise?

- Can arise naturally (measurement error): E.g. blood pressure, particulate histogram, truncation/rounding.
- 'Big Data' context:
  - Symbolic data points can summarise a complex & very large dataset in a compact manner.
  - Retaining maximal relevant information in original dataset.
  - Collapse over data not needed in detail for analysis.
  - Summarised data have own internal structure, which must be taken into account in any analysis.

Big data → small (symb) data
Easier to analyse (hopefully!)

Possible use in data privacy?
Individual can’t be identified.

Statistical question:
How to do statistical analysis for this form of data?
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How to analyse symbolic data?

State of the art:

**Poorly developed** in terms of inferential methods.

**Current approaches:**
- Descriptive statistics (means, covariances)
  ⇒ Methods based on $1^{st}/2^{nd}$ moments: clustering, PCA etc.
- Ad-hoc approaches (e.g. regression)
  ⇒ Can be plain wrong for inference/prediction.
- Single technique for constructing likelihood functions
  ⇒ Limited model-based inferences

**Over-prevalence of models for intervals & assuming uniformity**
⇒ Need to move beyond uniformity (Lynne Billard)

**Current SDA research:**
Developing practical model-based (e.g. likelihood-based) procedures for statistical inference using symbolic data for general symbols.
Symbol: $S = (S^1, \ldots, S^n)^\top$

E.g. For random intervals $[a_i, b_i], \ i = 1, \ldots, n$:

- $S_i = (a_i, b_i)^\top$
- $S_i = (m_i, \log r_i)^\top$

Then specify a standard (classical data) model for $S_1, \ldots, S_n$. E.g.

$$(m_i, \log r_i)^\top \sim N(\mu, \Sigma)$$

**Model specification issues:**

- Need to find credible models for general $S$
  - Not always obvious how to do this.
  - Easy to specify models for classical data (e.g. GEV).
  - How to develop models for symbols (with internal variation)?
  - Can’t just fit to means. How to account for variation? etc.
Existing and new SDA models

Existing models for symbols (2) (Le Rademacher & Billard, 2011)

Inference issues:

- Symbol are summaries of classical data
  - Inference at symbol level only
- Ok but what if interest in modelling underlying data?
  - Want full distributional predictions of \( x \) (not just mean/var)

Symbol issues:

- Symbol assumptions are sometimes unrealistic
  - Distribution with the interval \([a, b]\) often assumed uniform.
  - Extremely unlikely and affects inference/prediction.
- Symbol parametrisation are not always stable
  - E.g. \([a, b] = (m, \log r)^T\), when \(a \to b\) then \(\log r \to \infty\)

Q: How to fit models and make predictions at the level of the classical data, based on observed symbols?
One possible approach \cite{beranger2017}

**The general approach:**

\[ L(S|\theta, \phi) \propto \int_x g(S|x, \phi)L(x|\theta)dx \]

where

- \( L(x|\theta) \) – standard classical data likelihood
- \( g(S|x, \phi) \) – probability of obtaining \( S \) given classical data \( x \)
- \( L(S|\theta) \) – new symbolic likelihood for parameters of classical model

**Gist:** Fitting the standard classical model \( L(x|\theta) \), when the data are viewed only through symbols \( S \) as summaries.

**Limiting case:** as \( S_i \rightarrow x_i \), then \( g(S_i|x, \phi) \rightarrow g(x_i|x) = \delta_{x_i}(x) \) and so

\[ L(S_i|\theta, \phi) \propto \int_x \delta_{x_i}(x)L(x|\theta)dx = L(x_i|\theta) \quad \text{(classical likelihood)} \]

- Different symbols give different \( g(S|x, \phi) \) (and \( \therefore L(S|\theta, \phi) \)).
How to construct $g(S|x, \phi)$?

- Typically we can easily describe the distribution of $X|S$:
  - **Intervals**: $x \sim U(a, b)$ where $S = (a, b)^T$
  - **Histograms**: $x \sim \begin{cases} w_i U(b_i, b_{i+1}) & b_i \leq x \leq b_{i+1} \\ 0 & \text{else} \end{cases}$ for fixed $\{b_i\}$ where $S = (s_1, \ldots, s_B)^T$, $w_i = s_i / \sum_k s_k$.
  - **Gaussian**: $x \sim N(\mu, \Sigma)$ where $S = (\mu, \Sigma)^T$.

- Although $U(a, b)$ specifications are unrealistic (we avoid this later).

- If we specify a prior/marginal on $S$, we then obtain
  
  $$g(S|x, \phi) = g(S|x) = \frac{f(x|S)f(S)}{f(x)}$$

  where $f(x) = \int f(x|S)f(S)dx$.

- Cute for Bayesians: use a posterior to build a classical likelihood :-)
Specific cases (1)

- **Example (1)**: No specified generative model $L(x|\theta)$

$$L(S|\theta, \phi) \propto \int_x g(S|x, \phi)L(x|\theta)dx$$

$$\Rightarrow L(S|\phi) \propto g(S|\phi)$$

That is:

Directly modelling symbol = existing likelihood approach

(Le Rademacher & Billard, 2011) ✔
Specific cases (2): Random intervals

Example (2): Random intervals: \( S = (S_\ell, S_u)^\top \)

Assume:
- \( X_1, \ldots, X_n \sim h(x|\omega) \) for some \( h \) (not uniform!) and
- \( S_\ell = X(\ell) \) and \( S_u = X(u) \) are lower/upper order statistics.

Then density of \( X|S \) is easily specified as:

\[
f(x|s_\ell, s_u) = \prod_{k=1}^n h^{(s_\ell)}(x_k|\omega) \prod_{k=1}^n h^{(s_\ell, s_u)}(x_k|\omega) \prod_{k=1}^n h^{(s_u)}(x_k|\omega) \delta_{s_\ell}(x(\ell)) \delta_{s_u}(x(u))
\]

where
- \( x = (x(1), \ldots, x(n))^\top \)
- \( h^{(s_\ell)}(x|\omega) = h(x|\omega) / H(s_\ell|\omega) I(x < s_\ell) \),
- \( h^{(s_u)}(x|\omega) = h(x|\omega) / (1 - H(s_u|\omega)) I(x > s_u) \),
- \( h^{(s_\ell, s_u)}(x|\omega) = h(x|\omega) / (H(s_u|\omega) - H(s_\ell|\omega)) I(s_\ell < x < s_u) \).
- Delta functions enforce \( x(\ell) = S_\ell \) and \( x(u) = S_u \).
Specific cases (2): Random intervals

Now, as $X_1, \ldots, X_n \sim h(x|\omega)$, we also have

$$f(s_\ell, s_u|\omega) = \frac{n!}{(\ell - 1)!(u - \ell - 1)!(n - u)!} H(s_\ell|\omega)^{\ell - 1}$$

$$\times [H(s_u|\omega) - H(s_\ell|\omega)]^{u - \ell - 1} [1 - H(s_u|\omega)]^{n - u} h(s_\ell|\omega)h(s_u|\omega)$$

where $H(x|\omega) = \int h(z|\omega)dz$.

And so we have the joint distribution as

$$f(x, s_\ell, s_u|\omega) = \frac{n!}{(\ell - 1)!(u - \ell - 1)!(n - u)!} \prod_{k=1}^{n} h(x_k|\omega)\delta_{s_\ell}(x(\ell))\delta_{s_u}(x(u))$$

and finally

$$g(s_\ell, s_u|x) = \frac{n!}{(\ell - 1)!(u - \ell - 1)!(n - u)!} \delta_{s_\ell}(x(\ell))\delta_{s_u}(x(u)).$$

**Note:** This is independent of the form of $h(x|\omega)$!
Specific cases (2): Random intervals

- Now if we want to fit the model $X_1, \ldots, X_n \sim g(x|\theta)$, this gives us

$$L(s_\ell, s_u|\theta) \propto \int_x g(s_\ell, s_u|x, \phi) \prod_{k=1}^n g(x_k|\theta) dx$$

$$\propto \frac{n!}{(\ell - 1)!(u - \ell - 1)!(n - u)!} G(s_\ell|\theta)^{\ell - 1} [G(s_u|\theta) - G(s_\ell|\theta)]^{u - \ell - 1}$$

$$\times [1 - G(s_u|\theta)]^{n - u} g(s_\ell|\theta) g(s_u|\theta)$$

where $G(x|\theta) = \int g(z|\theta) dz$

$\Rightarrow$ the (known) joint distribution of $(\ell, u)$-th order statistics of $\{X_k\}$. ✓

- When $S_\ell = \min_k X_k$ and $S_u = \max_k X_k$:

$$L(s_1, s_n|\theta) \propto n(n - 1) [G(s_n|\theta) - G(s_1|\theta)]^{n - 2} g(s_1|\theta) g(s_n|\theta), \quad s_1 < s_2$$

$\Rightarrow$ the (known) joint distribution of min/max of $\{X_k\}$. ✓

- Symbolic $\rightarrow$ Classical check:

If $S_\ell \rightarrow S_u = x$ (with $n = 1$) then $L(s_\ell, s_u|\theta) = g(x|\theta)$. ✓
Specific cases (3): Random histograms

- Underlying data
  \(X_1, \ldots, X_n \in \mathbb{R}^p \sim h(x|\omega).\)

- Collected into histogram (random counts) with fixed bins as:
  \[ S = (s_1, \ldots, s_B)^	op = (\#X_i \in B_1, \ldots, \#X_i \in B_B)^	op \]
  such that \(\sum_b s_b = n.\)

- The density of \(X|S\) is
  \[
  f(x|s) = \prod_{b=1}^B \prod_{\ell=1}^{s_b} h^{(b)}(x^\ell_b|\omega) I(x^\ell_b \in B_b)
  \]
  where
  - \(x^\ell_b\) is the \(\ell\)-th observation in bin \(B_b.\)
  - \(h^{(b)}(x|\omega) \propto h(x|\omega) I(x \in B_b).\)
  - Enforces \(s_b\) observations in bin \(B_b.\)
Specific cases (3): Random histograms

- By construction the (prior) distribution of counts $S = (s_1, \ldots, s_B)^T$ is

$$f(S|\omega) = \frac{n!}{s_1! \ldots s_B!} \prod P^h_{b}(\omega)^{s_b}$$

where

$$P^h_{b}(\omega) = \int_{B_b} h(x|\omega) dx$$

is the probability that any $x$ will fall in bin $B_b$.

- Consequently

$$f(x, S|\omega) = \frac{n!}{s_1! \ldots s_n!} \prod_{i=1}^{n} h(x_i|\omega) \prod_{b=1}^{B} l \left( \sum_{i=1}^{n} l(x_i \in B_b) = s_b \right)$$
Specific cases (3): Random histograms

As a result

\[ g(S|x) = \frac{n!}{s_1! \ldots s_B!} \prod_{b=1}^{B} \left( \sum_{i=1}^{n} I(x_i \in B_b) = s_b \right) . \]

Now if we want to fit the model \( X_1, \ldots, X_n \sim g(x|\theta) \), this gives us

\[ L(S|\theta) \propto \int_x g(S|x) \prod_{k=1}^{n} g(x_k|\theta) dx \]

\[ \propto \frac{n!}{s_1! \ldots s_n!} \prod_{b=1}^{B} [P_b^g(\theta)]^{s_b} \]

where \( P_b^g(\theta) = \int_{B_b} g(x|\theta) dx \)

\[ \Rightarrow \text{generalises univariate result of McLachlan & Jones (1988). } \checkmark \]
Specific cases (3): Random histograms

- **Limiting case:** recover classical likelihood as $B \to \infty$

\[
\lim_{B \to \infty} L(S|\theta) \propto \lim_{B \to \infty} \frac{n!}{s_1! \ldots s_B!} \prod_{b=1}^{B} \left[ \int_{B_b} g(z|\theta) \, dz \right]^{s_b} = L(X_1, \ldots, X_n|\theta)
\]

⇒ recover classical analysis as we approach classical data. ✓

- **Consistency:** Can show that with a sufficient number of histogram bins can perform analysis arbitrarily close to analysis with full dataset.

- **Computationally scalable:** Working with counts not computationally expensive latent data.

- **Some approximation of** $L(S|\theta)$ **to** $L(x|\theta)$ depending on level of discretisation. Work needed to quantify this.

- More complicated if data are not iid but exchangeable (Zhang & Sisson, in preparation)
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Motivation

QUESTION: What is the expected maximum temperature across some region within the next 50 or 100 years?

Figure: Heat wave in South East Australia (January 2017)
Motivation

What do we know?
- Environmental extremes are spatial ⇒ SPATIAL EXTREMES
- Max-stable processes are a convenient tool

Drawbacks and challenges?
- High dimensional distributions not always available, computationally costly ⇒ Composite likelihood (Padoan et al. 2010)
- Unfeasible for a large number of locations and temporal observations

**PROPOSAL:** use Symbolic Data Analysis (SDA)
Max-stable processes

- **Definition:** Let $X_1, X_2, \ldots$, be i.i.d replicates of $X(s), s \in S \subset \mathbb{R}^d$. $Y(s)$ is a max-stable process if $\exists a_n(s) > 0$ and $b_n(s)$, continuous such that

$$\left\{ \max_{i=1,\ldots,n} \frac{X_i(s) - b_n(s)}{a_n(s)} \right\}_{s \in S} \xrightarrow{d} \{Y(s)\}_{s \in S}.$$

- Spectral representation (de Haan, 1984; Schlather, 2002) $\Rightarrow$ Max-stable models

**Gaussian extreme value model (Smith, 1990)** defined by

$$Y(s) = \max_{1 \leq i} \{ \zeta_i \phi_d(s; t_i, \Sigma) \}, s \in \mathbb{R}^d$$

where $(\zeta_i, t_i)_{1 \leq i}$ are the points of a point process on $(0, \infty) \times \mathbb{R}^d$.

For $d = 2$, the bivariate cdf of $(Y(s_1), Y(s_2)), s_1, s_2 \in \mathbb{R}^2$ is

$$P(Y(s_1) \leq y_1, Y(s_2) \leq y_2) = \exp \left( -\frac{1}{v_1} \Phi \left( \frac{a}{2} + \frac{1}{a} \log \frac{v_2}{v_1} \right) - \frac{1}{v_2} \Phi \left( \frac{a}{2} + \frac{1}{a} \log \frac{v_1}{v_2} \right) \right),$$

where $v_i = \left( 1 - \xi_i \frac{\nu_i - \mu_i}{\sigma_i} \right)^{-\frac{1}{\xi}}, i = 1, 2$ and $a^2 = (z_1 - z_2)^T \Sigma^{-1} (z_1 - z_2)$. 
An example in EVT
Composite Likelihood

**Composite Likelihood (1)**

- Let \( \mathbf{X} = (X_1, \ldots, X_N) \) denote a vector of \( N \) i.i.d. rv's taking values in \( \mathbb{R}^K \) with realisation \( \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{K \times N} \) and density function \( g_{\mathbf{X}}(\cdot; \theta) \).

- Define a subset of \( \{1, \ldots, K\} \) by \( \mathbf{i} = (i_1, \ldots, i_j) \), where \( i_1 < \cdots < i_j \) with \( i_j \in \{1, \ldots, K\} \) for \( j = 1, \ldots, K - 1 \).

- Then for \( n = 1, \ldots, N \), \( x_n^i \in \mathbb{R}^j \) defines a subset of \( x_n \) and \( \mathbf{x}^i = (x_1^i, \ldots, x_N^i) \in \mathbb{R}^{j \times N} \), defines a subset of \( \mathbf{x} \).

The **\( j \)-wise composite likelihood function**, \( \text{CL}^{(j)} \), is given by

\[
L_{\text{CL}}^{(j)}(\mathbf{x}; \theta) = \prod_i g_{\mathbf{x}^i}(\mathbf{x}^i; \theta),
\]

where \( g_{\mathbf{x}^i} \) is a \( j \)-dimensional likelihood function.
When $j = 2$, the \textit{pairwise} composite log-likelihood function, $l^{(2)}_{\text{CL}}$ is given by

$$l^{(2)}_{\text{CL}}(x; \theta) = \sum_{i_1=1}^{K-1} \sum_{i_2=i_1+1}^{K} \log g_{X_i}(x^{i_1}, x^{i_2}; \theta) \Rightarrow \frac{NK(K-1)}{2} \text{ terms}$$

The resulting \textbf{maximum j-wise composite likelihood estimator} $\hat{\theta}_{\text{CL}}^{(j)}$ is asymptotically consistent and distributed as

$$\sqrt{N} \left( \hat{\theta}_{\text{CL}}^{(j)} - \theta \right) \to \mathcal{N}(0, G(\theta)^{-1}) ,$$

where $G(\theta) = H(\theta)J(\theta)^{-1}H(\theta)$, $J(\theta) = \text{var}(\nabla l^{(j)}_{\text{CL}}(\theta))$ is a variability matrix and $H(\theta) = -\mathbb{E}(\nabla^2 l^{(j)}_{\text{CL}}(\theta))$ is a sensitivity matrix.
Consider we are only interested in a subset of size $j$ of the $K$ dimensions.

Let $b^i$ be the subset of $b$ defining the coordinates of a $j$—dimensional histogram bin and let $B^i = (B^i_1, \ldots, B^i_j)$ be the vector of the number of marginal bins.

The symbolic likelihood function associated with the vector of counts $s^i_j = (s^i_{b^i_1}, \ldots, s^i_{b^i_{B^i_j}})$ of length $B^i_1 \times \cdots \times B^i_j$ is

$$L(s^i_j; \theta) = \frac{N!}{s^i_{b^i_1}! \cdots s^i_{b^i_{B^i_j}}!} \prod_{b^i_1=1}^{B^i_1} P_{b^i}(\theta)^{s^i_{b^i_1}},$$

where $P_{b^i}(\theta) = \int_{\gamma^i_{b^i_1}} \cdots \int_{\gamma^i_{b^i_{B^i_j}}} g_X(x; \theta) dx$ and $g_X$ is a $j$—dim density.
Histogram-valued symbols (2)

- $s_j = \{s^i_{jt}; t = 1, \ldots, T, i = (i_1, \ldots, i_j), i_1 < \ldots < i_j\}$ represents the set of $j$-dimensional observed histograms for the symbolic-valued random variable $S_j$

- The symbolic $j$-wise composite likelihood function ($SCL^{(j)}$) is given by

$$L_{SCL}^{(j)}(s_j; \theta) = \prod_{t=1}^{T} \prod_{i} L(s^i_{jt}; \theta)$$

- Components of the Godambe matrix are given by

$$\hat{H}(\hat{\theta}_{SCL}^{(j)}) = -\frac{1}{N} \sum_{t=1}^{T} \sum_{i} \nabla^2 l(s^i_{jt}; \hat{\theta}_{SCL}^{(j)})$$

$$\hat{J}(\hat{\theta}_{SCL}^{(j)}) = \frac{1}{N} \sum_{t=1}^{T} \left( \sum_{i} \nabla l(s^i_{jt}; \hat{\theta}_{SCL}^{(j)}) \right) \left( \sum_{i} \nabla l(s^i_{jt}; \hat{\theta}_{SCL}^{(j)}) \right)^T$$
Simulation experiments: the set up

- \( K \) locations are generated uniformly on a \((0, 40) \times (0, 40)\) grid
- \( N \) realisations of the Smith model are generated for each location
- MLE’s are obtained using \( \text{CL}^{(2)} \) and \( \text{SCL}^{(2)} \)
Experiment 1 - Increasing the number of bins

- $N = 1000$, $K = 15$, $T = 1$, $\Sigma = \begin{bmatrix} 300 & 0 \\ 0 & 300 \end{bmatrix}$, Repetitions = 1000

Figure: Mean of MLEs for $\theta = (\sigma_{11}, \sigma_{12}, \sigma_{22}, \mu, \sigma, \xi)$ using $\text{CL}^{(2)}$ and $\text{SCL}^{(2)}$, for increasing number of bins in bivariate histograms.
**Experiment 2 - Computation time**

- $B = 25$, $K = 10, 100$, $T = 1$, Repetitions = 10

<table>
<thead>
<tr>
<th>$N$</th>
<th>$K = 10$</th>
<th>$K = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_c$</td>
<td>$t_s$</td>
</tr>
<tr>
<td>100</td>
<td>9.8</td>
<td>18.6</td>
</tr>
<tr>
<td>500</td>
<td>27.6</td>
<td>26.2</td>
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<td>50000</td>
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<td>100000</td>
<td>5610.7</td>
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</tr>
<tr>
<td>500000</td>
<td>31083.1</td>
<td>23.2</td>
</tr>
</tbody>
</table>

**Table:** Mean computation times (sec) to optimise the regular and symbolic composite likelihood ($t_c$ and $t_s$), and to aggregate the data into bivariate histograms ($t_{hist}$)
**Experiement 3 - Convergence of variances (1)**

- $B = 25$, $N = 1000$, $K = 10$, Number of repetitions = 1000

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\sigma_{11}$</th>
<th>$\sigma_{12}$</th>
<th>$\sigma_{22}$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>226.93</td>
<td>97.63</td>
<td>167.27</td>
<td>0.105</td>
<td>0.051</td>
<td>0.030</td>
</tr>
<tr>
<td>5</td>
<td>203.04</td>
<td>87.36</td>
<td>149.66</td>
<td>0.095</td>
<td>0.047</td>
<td>0.028</td>
</tr>
<tr>
<td>10</td>
<td>143.92</td>
<td>61.95</td>
<td>106.04</td>
<td>0.071</td>
<td>0.036</td>
<td>0.021</td>
</tr>
<tr>
<td>20</td>
<td>102.23</td>
<td>44.04</td>
<td>75.27</td>
<td>0.054</td>
<td>0.029</td>
<td>0.016</td>
</tr>
<tr>
<td>40</td>
<td>72.93</td>
<td>31.48</td>
<td>53.64</td>
<td>0.043</td>
<td>0.024</td>
<td>0.013</td>
</tr>
<tr>
<td>50</td>
<td>65.52</td>
<td>28.31</td>
<td>48.16</td>
<td>0.040</td>
<td>0.023</td>
<td>0.012</td>
</tr>
<tr>
<td>100</td>
<td>47.38</td>
<td>20.55</td>
<td>34.71</td>
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<td>0.020</td>
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</tr>
<tr>
<td>200</td>
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<td>1000</td>
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<td>13.11</td>
<td>0.025</td>
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</tr>
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<td>Classic</td>
<td>16.65</td>
<td>10.53</td>
<td>10.69</td>
<td>0.020</td>
<td>0.014</td>
<td>0.009</td>
</tr>
</tbody>
</table>

Table: Mean variances calculated from $\text{CL}^{(2)}$ and $\text{SCL}^{(2)}$ for $\theta = (\sigma_{11}, \sigma_{12}, \sigma_{22}, \mu, \sigma, \xi)$ for increasing $T$. 

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First steps in SDA  
October 4, 2017
Experiment 3 - Convergence of variances (2)

- \( \hat{J}(\hat{\theta}_S^{(j)}) \) requires \( T \to N \) and \( B \to \infty \) for the convergence towards the classical Godambe matrices to occur.

- For \( T \) fixed, convergence still occurs as \( B \to \infty \) towards a different expression.

Figure: Mean variances calculated from SCL\(^{(2)}\) for fixed \( T \) and increasing \( B \).
Real data analysis: an overview

- Maximum temperatures across Australia

- **Data:**
  - Focus on fortnightly maxima at \( K = 105 \) locations over summer months
  - 3 sets: historical \((N = 970)\), RCP4.5 and RCP8.5 (both \( N = 540 \))

- Bivariate histograms are constructed for all pairs of locations for \( B = 15, 20, 25, 30 \).

**Figure:** Study region
Model fitting

Fit the Smith model with mean and variance parameters as linear functions of space

<table>
<thead>
<tr>
<th>( B )</th>
<th>( \sigma_{11} )</th>
<th>( \sigma_{12} )</th>
<th>( \sigma_{22} )</th>
<th>( \xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Historical Data</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>176.4 (0.285)</td>
<td>-28.7 (0.032)</td>
<td>76.8 (0.329)</td>
<td>-0.266 (0.053)</td>
</tr>
<tr>
<td>20</td>
<td>164.2 (0.289)</td>
<td>-29.3 (0.030)</td>
<td>74.3 (0.469)</td>
<td>-0.264 (0.049)</td>
</tr>
<tr>
<td>25</td>
<td>162.4 (0.217)</td>
<td>-29.9 (0.033)</td>
<td>75.3 (0.284)</td>
<td>-0.264 (0.049)</td>
</tr>
<tr>
<td>30</td>
<td>161.6 (0.201)</td>
<td>-32.3 (0.029)</td>
<td>74.4 (0.234)</td>
<td>-0.264 (0.050)</td>
</tr>
<tr>
<td>RCP4.5 Data</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>160.9 (0.942)</td>
<td>-34.1 (0.083)</td>
<td>79.0 (0.222)</td>
<td>-0.249 (0.074)</td>
</tr>
<tr>
<td>20</td>
<td>163.5 (0.595)</td>
<td>-41.1 (0.073)</td>
<td>77.6 (0.245)</td>
<td>-0.249 (0.076)</td>
</tr>
<tr>
<td>25</td>
<td>150.3 (0.349)</td>
<td>-33.1 (0.065)</td>
<td>70.7 (0.170)</td>
<td>-0.250 (0.073)</td>
</tr>
<tr>
<td>30</td>
<td>150.2 (0.150)</td>
<td>-31.6 (0.024)</td>
<td>70.7 (0.154)</td>
<td>-0.250 (0.069)</td>
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<tr>
<td>RCP8.5 Data</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>128.7 (0.860)</td>
<td>-19.6 (0.092)</td>
<td>67.7 (0.392)</td>
<td>-0.232 (0.061)</td>
</tr>
<tr>
<td>20</td>
<td>128.0 (0.630)</td>
<td>-19.6 (0.129)</td>
<td>66.6 (0.332)</td>
<td>-0.231 (0.059)</td>
</tr>
<tr>
<td>25</td>
<td>136.0 (0.395)</td>
<td>-15.1 (0.093)</td>
<td>59.4 (0.317)</td>
<td>-0.234 (0.060)</td>
</tr>
<tr>
<td>30</td>
<td>129.9 (0.401)</td>
<td>-13.6 (0.083)</td>
<td>56.4 (0.294)</td>
<td>-0.233 (0.055)</td>
</tr>
</tbody>
</table>

Figure: MLEs using the \( SCL(2) \) for various values of \( B \).
Estimated location parameter

**Figure:** Estimated surfaces for the location parameter using the $l^{(2)}_{SCL}$ function (left) and marginal GEV estimations (right)
Examples of return level plots

Figure: Estimated 95 year return levels using the $I_{SCL}^{(2)}$ function (left) and observed 95 year return levels (right)
Outline

1. What is Symbolic Data Analysis?
2. Existing and new SDA models
3. An example in EVT
4. Discussion
**Summary**

- **Completely new approach to SDA:**
  - Based on fitting underlying (classical) model ⇒ Much better!
  - View latent (classical) data through symbols
  - **Recovers existing models** for symbols but is more general
  - **Recovers classical model** as $S \rightarrow x$
  - Works for more general symbols than currently in use
  - Illustration of practical use in extremes

- **Working on:**
  - Characterise trade-off between accuracy and computation
  - Finalising procedure for **distribution valued symbols** (Gaussians, etc.)
  - **Design** symbols for best performance

**THANK YOU!**